

# A CHARACTERIZATION OF THE MARTINGALE PROPERTY OF EXPONENTIALLY AFFINE PROCESSES

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**ABSTRACT.** We consider local martingales of exponential form  $M = e^X$  or  $\mathcal{E}(X)$  where  $X$  denotes one component of a multivariate affine process in the sense of Duffie, Filipović and Schachermayer [8]. By completing the characterization of conservative affine processes in [8, Section 9], we provide deterministic necessary and sufficient conditions in terms of the parameters of  $X$  for  $M$  to be a true martingale.

## 1. INTRODUCTION

A classical question in probability theory comprises the following. Suppose the ordinary resp. stochastic exponential  $M = \exp(X)$  resp.  $\mathcal{E}(X)$ <sup>1</sup> of some process  $X$  is a positive *local martingale* and hence a supermartingale. Then under what (if any) additional assumptions is it in fact a *true martingale*?

This seemingly technical question is of considerable interest in diverse applications, for example, absolute continuity of distributions of stochastic processes (cf., e.g., [3] and the references therein), absence of arbitrage in financial models (see, e.g., [6]) or verification of optimality in stochastic control (cf., e.g., [9]).

In a general semimartingale setting it has been shown in [11] that any supermartingale  $M$  is a martingale if and only if it is non-explosive under the associated *Föllmer measure* (also cf. [26]). However, this general result is hard to apply in concrete models, since it is expressed in purely probabilistic terms. Consequently, there has been extensive research focused on exploiting the link between martingales and non-explosion in various more specific settings, see, e.g., [25]. In particular, *deterministic* necessary and sufficient conditions for the martingale property

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<sup>1</sup>The *stochastic exponential*  $\mathcal{E}(X)$  of a semimartingale  $X$  is the unique solution of the linear SDE  $d\mathcal{E}(X)_t = \mathcal{E}(X)_{t-} dX_t$  with  $\mathcal{E}(X)_0 = 1$ , cf., e.g., [13, I.4.61] for more details.

of  $M$  have been obtained if  $X$  is a one-dimensional diffusion (cf., e.g., [7, 2] and the references therein; also compare [21]).

For processes with jumps, the literature is more limited and mostly focused on sufficient criteria as in [20, 16, 22, 15]. By the independence of increments and the Lévy-Khintchine formula, no extra assumptions are needed for  $M$  to be a true martingale if  $X$  is a Lévy process. For the more general class of *affine processes* characterized in [8] the situation becomes more involved. While no additional conditions are needed for continuous affine processes, this no longer remains true in the presence of jumps (cf. [15, Example 3.11]). In this situation a necessary and sufficient condition for one-factor models has been established in [18, Theorem 2.5], whereas easy-to-check sufficient conditions for the general case are provided by [15, Theorem 3.1].

In the present study, we complement these results by sharpening [15, Theorem 3.1] in order to provide deterministic necessary and sufficient conditions for the martingale property of  $M = \mathcal{E}(X^i)$  resp.  $\exp(X^i)$  in the case where  $X^i$  is one component of a general non-explosive affine process  $X$ . As in [18, 15] these conditions are expressed in terms of the admissible *parameters* which characterize the distribution of  $X$  (cf. [8]).

Since we also use the linkage to non-explosion, we first complete the characterization of *conservative*, i.e. non-explosive, affine processes from [8, Section 9]. Generalizing the arguments from [15], we then establish that  $M$  is a true martingale if and only if it is a local martingale and a related affine process is conservative. Combined with the characterization of local martingales in terms of semimartingale characteristics [14, Lemma 3.1] this then yields necessary and sufficient conditions for the martingale property of  $M$ .

The article is organized as follows. In Section 2, we recall terminology and results on affine Markov processes from [8]. Afterwards, we characterize conservative affine processes. Subsequently, in Section 4, this characterization is used to provide necessary and sufficient conditions for the martingale property of exponentially affine processes. Appendix A develops ODE comparison results in a general non-Lipschitz setting that are used to establish the results in Section 3.

## 2. AFFINE PROCESSES

For stochastic background and terminology, we refer to [13, 23]. We work in the setup of [8], that is we consider a time-homogeneous Markov process with state space  $D := \mathbb{R}_+^m \times \mathbb{R}^n$ , where  $m, n \geq 0$  and  $d = m + n \geq 1$ . We write  $p_t(x, d\xi)$  for its transition function and let  $(X, \mathbb{P}_x)_{x \in D}$  denote its realization on the canonical filtered space  $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t \in \mathbb{R}_+})$  of paths  $\omega : \mathbb{R}_+ \rightarrow D_\Delta$  (the one-point-compactification of  $D$ ). For every  $x \in D$ ,  $\mathbb{P}_x$  is a probability measure on  $(\Omega, \mathcal{F}^0)$  such that  $\mathbb{P}_x(X_0 = x) = 1$  and the Markov property holds, i.e.

$$\begin{aligned} \mathbb{E}_x(f(X_{t+s}) | \mathcal{F}_s^0) &= \int_D f(\xi) p_t(X_s, d\xi) \\ &= \mathbb{E}_{X_s}(f(X_t)), \quad \mathbb{P}_x\text{-a.s.} \quad \forall t, s \in \mathbb{R}_+, \end{aligned}$$

for all bounded Borel-measurable functions  $f : D \rightarrow \mathbb{C}$ . The Markov process  $(X, \mathbb{P}_x)_{x \in D}$  is called *conservative* if  $p_t(x, D) = 1$ , *stochastically continuous* if we have  $p_s(x, \cdot) \rightarrow p_t(x, \cdot)$  weakly on  $D$ , for  $s \rightarrow t$ , for every  $(t, x) \in \mathbb{R}_+ \times D$ , and *affine* if, for every  $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$ , the characteristic function of  $p_t(x, \cdot)$  is of the form

$$\int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = \exp(\psi_0(t, u) + \langle \psi(t, u), x \rangle), \quad \forall x \in D, \quad (2.1)$$

for some  $\psi_0(t, u) \in \mathbb{C}$  and  $\psi(t, u) = (\psi_1(t, u), \dots, \psi_d(t, u)) \in \mathbb{C}^d$ . Note that  $\psi(t, u)$  is uniquely specified by (2.1). But  $\text{Im}(\psi_0(t, u))$  is only determined up to multiples of  $2\pi$ . As usual in the literature, we enforce uniqueness by requiring the continuity of  $u \mapsto \psi_0(t, u)$  as well as  $\psi_0(t, 0) = \log(p_t(0, D)) \in (-\infty, 0]$  (cf., e.g., [1, §26]).

For every stochastically continuous affine process, the mappings  $(t, u) \mapsto \psi_0(t, u)$  and  $(t, u) \mapsto \psi(t, u)$  can be characterized in terms of the following quantities:

**Definition 2.1.** Denote by  $h = (h_1, \dots, h_d)$  the truncation function on  $\mathbb{R}^d$  defined by

$$h_k(\xi) := \begin{cases} 0, & \text{if } \xi_k = 0, \\ (1 \wedge |\xi_k|) \frac{\xi_k}{|\xi_k|}, & \text{otherwise.} \end{cases}$$

Parameters  $(\alpha, \beta, \gamma, \kappa)$  are called *admissible*, if

- $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$  with symmetric positive semi-definite  $d \times d$ -matrices  $\alpha_j$  such that  $\alpha_j = 0$  for  $j \geq m+1$  and  $\alpha_j^{kl} = 0$  for  $0 \leq j \leq m$ ,  $1 \leq k, l \leq m$  unless  $k = l = j$ ;
- $\kappa = (\kappa_0, \kappa_1, \dots, \kappa_d)$  where  $\kappa_j$  is a Borel measure on  $D \setminus \{0\}$  such that  $\kappa_j = 0$  for  $j \geq m+1$  as well as  $\int_{D \setminus \{0\}} \|h(\xi)\|^2 \kappa_j(d\xi) < \infty$  for  $0 \leq j \leq m$  and

$$\int_{D \setminus \{0\}} |h_k(\xi)| \kappa_j(d\xi) < \infty, \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j;$$

- $\beta = (\beta_0, \beta_1, \dots, \beta_d)$  with  $\beta_j \in \mathbb{R}^d$  such that  $\beta_j^k = 0$  for  $j \geq m+1$ ,  $1 \leq k \leq m$  and

$$\beta_j^k - \int_{D \setminus \{0\}} h_k(\xi) \kappa_j(d\xi) \geq 0, \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j.$$

- $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_d)$ , where  $\gamma_j \in \mathbb{R}_+$  and  $\gamma_j = 0$  for  $j = m+1, \dots, d$ .

Affine Markov processes and admissible parameters are related as follows (cf. [8, Theorem 2.7] and [19, Theorem 5.1]):

**Theorem 2.2.** Let  $(X, \mathbb{P}_x)_{x \in D}$  be a stochastically continuous affine process. Then there exist admissible parameters  $(\alpha, \beta, \gamma, \kappa)$  such that  $\psi_0(t, u)$  and  $\psi(t, u)$  are given as solutions to the generalized Riccati equations

$$\partial_t \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = u, \quad (2.2)$$

$$\partial_t \psi_0(t, u) = R_0(\psi(t, u)), \quad \psi_0(0, u) = 0, \quad (2.3)$$

where  $R = (R_1, \dots, R_d)$  and for  $0 \leq i \leq d$ ,

$$R_i(u) := \frac{1}{2} \langle \alpha_i u, u \rangle + \langle \beta_i, u \rangle - \gamma_i + \int_{D \setminus \{0\}} (e^{\langle u, \xi \rangle} - 1 - \langle u, h(\xi) \rangle) \kappa_i(d\xi). \quad (2.4)$$

Conversely, for any set  $(\alpha, \beta, \gamma, \kappa)$  of admissible parameters there exists a unique stochastically continuous affine process such that (2.1) holds for all  $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$ , where  $\psi_0$  and  $\psi$  are given by (2.3) and (2.2).

Since any stochastically continuous affine process  $(X, \mathbb{P}_x)_{x \in D}$  is a Feller process (cf. [8, Theorem 2.7]), it admits a càdlàg modification and hence can be realized on the space of càdlàg paths  $\omega : \mathbb{R}_+ \rightarrow D_\Delta$ . If  $(X, \mathbb{P}_x)_{x \in D}$  is also conservative it turns out to be a semimartingale in the usual sense and hence can be realized on the Skorokhod space  $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}_t^d)_{t \in \mathbb{R}_+})$  of  $D$ - rather than  $D_\Delta$ -valued càdlàg paths. Here,  $\mathcal{D}_t^d = \bigcap_{s > t} \mathcal{D}_s^{0,d}$  for the filtration  $(\mathcal{D}_t^{0,d})_{t \in \mathbb{R}_+}$  generated by  $X$ . The semimartingale characteristics of  $(X, \mathbb{P}_x)_{x \in D}$  are then given in terms of the admissible parameters:

**Theorem 2.3.** *Let  $(X, \mathbb{P}_x)_{x \in D}$  be a conservative, stochastically continuous affine process and let  $(\alpha, \beta, \gamma, \kappa)$  be the related admissible parameters. Then  $\gamma = 0$  and for any  $x \in D$ ,  $X = (X^1, \dots, X^d)$  is a semimartingale on  $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}_t^d)_{t \in \mathbb{R}_+}, \mathbb{P}_x)$  with characteristics  $(B, C, \nu)$  given by*

$$B_t = \int_0^t \left( \beta_0 + \sum_{j=1}^d \beta_j X_{s-}^j \right) ds, \quad (2.5)$$

$$C_t = \int_0^t \left( \alpha_0 + \sum_{j=1}^d \alpha_j X_{s-}^j \right) ds, \quad (2.6)$$

$$\nu(dt, d\xi) = \left( \kappa_0(d\xi) + \sum_{j=1}^d X_{t-}^j \kappa_j(d\xi) \right) dt, \quad (2.7)$$

relative to the truncation function  $h$ . Conversely, let  $X'$  be a  $D$ -valued semimartingale defined on some filtered probability space  $(\Omega', \mathcal{F}', (\mathcal{F}_t'), \mathbb{P}')$ . If  $\mathbb{P}'(X'_0 = x) = 1$  and  $X'$  admits characteristics of the form (2.5)-(2.7) with  $X_-$  replaced by  $X'_-$ , then  $\mathbb{P}' \circ X'^{-1} = \mathbb{P}_x$ .

*Proof.*  $\gamma = 0$  is shown in [8, Proposition 9.1]; the remaining assertions follow from [8, Theorem 2.12].  $\square$

### 3. CONSERVATIVE AFFINE PROCESSES

In view of Theorem 2.3, the powerful toolbox of semimartingale calculus is made available for affine processes, provided that the Markov process  $(X, \mathbb{P}_x)_{x \in D}$  is conservative. Hence, it is desirable to characterize this property in terms of the parameters of  $X$ . This is done in the present section. The main result is Theorem 3.4, which completes the discussion of conservativeness in [8, Section 9].

To prove this statement, we proceed as follows. First, we recall some properties of the generalized Riccati equations (2.2), (2.3) established by Duffie et al. [8]. In the crucial next step, we use the comparison results developed in the appendix to show that whereas the characteristic exponent  $\psi$  of the affine process  $X$  is not the *unique* solution to these equations in general, it is necessarily the *minimal* one among all such solutions. Using this observation, we can then show that conservativeness of the process  $X$  is indeed *equivalent* to uniqueness for the specific initial

value zero. Note that *sufficiency* of this uniqueness property was already observed in [8, Proposition 9.1]; here we show that this condition is also *necessary*.

Let us first introduce some definitions and notation. The partial order on  $\mathbb{R}^m$  induced by the natural cone  $\mathbb{R}_+^m$  is denoted by  $\leq$ . That is,  $x \leq 0$  if and only if  $x_i \leq 0$  for  $i = 1, \dots, m$ . A function  $g : D_g \rightarrow \mathbb{R}^m$  is *quasimonotone increasing* on  $D_g \subset \mathbb{R}^m$  (*qmi* in short, for a general definition see section A) if and only if for all  $x, y \in D_g$  and  $i = 1, \dots, m$  the following implication holds true:

$$(x \leq y, \quad x_i = y_i) \Rightarrow g_i(x) \leq g_i(y).$$

In the sequel we write  $\mathbb{R}_{--} := (-\infty, 0)$  and  $\mathbb{C}_{--} := \{c \in \mathbb{C} \mid \operatorname{Re}(c) \in \mathbb{R}_{--}\}$ . Moreover, we introduce the index set  $\mathcal{I} := \{1, \dots, m\}$  and, accordingly, define by  $u_{\mathcal{I}} = (u_1, \dots, u_m)$  the projection of the  $d$ -dimensional vector  $u$  onto the first  $m$  coordinates. Similarly  $R_{\mathcal{I}}$  denotes the first  $m$  components of  $R$ , i.e.  $R_{\mathcal{I}} = (R_1, \dots, R_m)$  and  $R_{\mathcal{I}}(u_{\mathcal{I}}, 0) := (R_1(u_1, \dots, u_m, 0, \dots, 0), \dots, R_m(u_1, \dots, u_m, 0, \dots, 0))$ . Finally,  $\psi_{\mathcal{I}}$  and  $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$  are defined analogously.

For this section the uniqueness of solutions to eqs. (2.2)–(2.3) is essential. It is addressed in the following remark. For more detailed information, we refer to [8, Sections 5 and 6].

- Remark 3.1.** (i) Due to the admissibility conditions on the jump parameters  $\kappa$  the domains of  $R_0$  and  $R$  can be extended from  $i\mathbb{R}^d$  to  $\mathbb{C}_{--}^m \times i\mathbb{R}^n$ . Moreover,  $R_0, R$  are analytic functions on  $\mathbb{C}_{--}^m \times i\mathbb{R}^n$ , and admit a unique continuous extension to  $\mathbb{C}_{--}^m \times i\mathbb{R}^n$ .
- (ii) In general,  $R$  is not locally Lipschitz on  $i\mathbb{R}^d$ , but only continuous (see [8, Example 9.3]). This lack of regularity prohibits to provide well-defined  $\psi_0, \psi$  by simply solving (2.2)–(2.3), because unique solutions do not always exist, again cf. [8, Example 9.3]. Hence another approach to construct unique characteristic exponents  $\psi_0, \psi$  is required. Duffie et al. [8] tackle this problem by first proving the existence of unique global solutions  $\psi_0^\circ, \psi^\circ$  on  $\mathbb{C}_{--}^m \times i\mathbb{R}^n$ , where uniqueness is guaranteed by the analyticity of  $R$ , see (i). Their unique continuous extensions to the closure  $\mathbb{C}_{--}^m \times i\mathbb{R}^n$  are also differentiable and solve (2.2)–(2.3) for  $u \in i\mathbb{R}^d$ . Moreover, they satisfy (2.1). Henceforth,  $\psi_0, \psi$  therefore denote these unique extensions.

**Lemma 3.2.** *The affine transform formula (2.1) also holds for  $u = (u_{\mathcal{I}}, 0) \in \mathbb{R}^d$  with characteristic exponents  $\psi_0(t, (u_{\mathcal{I}}, 0)) : \mathbb{R}_+ \times \mathbb{R}_{--}^m \rightarrow \mathbb{R}_{--}$  and  $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)) : \mathbb{R}_+ \times \mathbb{R}_{--}^m \rightarrow \mathbb{R}_{--}^m$  satisfying*

$$\partial_t \psi_0(t, (u_{\mathcal{I}}, 0)) = R_0((\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)), 0)), \quad \psi_0(0, (u_{\mathcal{I}}, 0)) = 0, \quad (3.1)$$

$$\partial_t \psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)) = R_{\mathcal{I}}((\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0)), 0)) \quad \psi_{\mathcal{I}}(0, (u_{\mathcal{I}}, 0)) = u_{\mathcal{I}}. \quad (3.2)$$

Furthermore we have:

- $R_0, R_{\mathcal{I}}$  are continuous functions on  $\mathbb{R}_{--}^m$  such that  $R_0(0) \leq 0, R_{\mathcal{I}}(0) \leq 0$
- $R_{\mathcal{I}}(u_{\mathcal{I}}, 0)$  is locally Lipschitz continuous on  $\mathbb{R}_{--}^m$  and *qmi* on  $\mathbb{R}_{--}^m$ ,
- $\psi_{\mathcal{I}}(t, (u_{\mathcal{I}}, 0))$  restricts to an  $\mathbb{R}_{--}^m$ -valued unique global solution  $\psi_{\mathcal{I}}^\circ(t, (u_{\mathcal{I}}, 0))$  of (3.2) on  $\mathbb{R}_+ \times \mathbb{R}_{--}^m$ .

*Proof.* By [19], any *stochastically continuous* affine processes is *regular* in the sense of [8]. Hence, the first statement is a consequence of [8, Proposition 6.4]. The regularity of  $R_0$  and  $R_I$  follows from [8, Lemma 5.3 (i) and (ii)]. Equation (2.4) shows  $R_0(0) \leq 0$  and  $R_I(0) \leq 0$ . The mapping  $v \mapsto R_I((v, 0))$  is qmi on  $\mathbb{R}_-^m$  by [17, Lemma 4.6], whereas the last assertion is stated in [8, Proposition 6.1].  $\square$

In the following crucial step we establish the minimality of  $\psi_I(t, (u_I, 0))$  among all solutions of (3.2) with respect to the partial order  $\leq$ .

**Proposition 3.3.** *Let  $T > 0$  and  $u_I \in \mathbb{R}_-^m$ . If  $g(t) : [0, T) \rightarrow \mathbb{R}_-^m$  is a solution of*

$$\partial_t g(t) = R_I(g(t), 0), \quad (3.3)$$

*subject to  $g(0) = u_I$ , then  $g(t) \geq \psi_I(t, (u_I, 0))$ , for all  $t < T$ .*

*Proof.* The properties of  $R_I$  established in Lemma 3.2 allow this conclusion by a use of Corollary A.3. For an application of the latter, we make the obvious choices  $f = R_I$ ,  $D_f = \mathbb{R}_-^m$ . Then we know that for  $u_I^\circ \in \mathbb{R}_-^m$  we have  $g(t) \geq \psi_I^\circ(t, (u_I, 0))$ , for all  $t < T$ . Now letting  $u_I^\circ \rightarrow u_I$  and using the continuity of  $\psi_I$  as asserted in Lemma 3.2 yields the assertion.  $\square$

We now state the main result of this section, which is a full characterization of conservative affine processes in terms of a uniqueness criterium imposed on solutions of the corresponding generalized Riccati equations. It is motivated by a partial result of this kind provided in [8, Proposition 9.1], which gives a necessary condition for conservativeness, as well as a sufficient one. Here, we show that their sufficient condition, which (modulo the assumption  $R(0) = 0$ ) equals (ii) below, is in fact also necessary for conservativeness. The proof is based on the comparison results for multivariate initial value problems developed in Appendix A.

**Theorem 3.4.** *The following statements are equivalent:*

- (i)  $(X, \mathbb{P}_x)_{x \in D}$  is conservative,
- (ii)  $R_0(0) = 0$  and there exists no non-trivial  $\mathbb{R}_-^m$ -valued local solution  $g(t)$  of (3.3) with  $g(0) = 0$ .

*Moreover, each of these statements implies that  $R(0) = 0$ .*

*Proof.* (i) $\Rightarrow$ (ii): By definition,  $X$  is conservative if and only if, for all  $t \geq 0$  and  $x \in D$ , we have

$$1 = p_t(x, D) = e^{\psi_0(t, 0) + \langle \psi(t, 0), x \rangle} = e^{\psi_0(t, 0) + \langle \psi_I(t, 0), x_I \rangle},$$

because  $\psi_i(t, (u_I, 0)) = 0$ , for  $i = m+1, \dots, d$ . By first putting  $x = 0$  and then using the arbitrariness of  $x$ , it follows that this is equivalent to

$$\psi_0(t, 0) = 0 \text{ and } \psi_I(t, 0) = 0, \quad \forall t \geq 0. \quad (3.4)$$

Let  $g$  be a (local) solution of (3.3) on some interval  $[0, T)$ , satisfying  $g(0) = 0$  and with values in  $\mathbb{R}_-^m$ . By Proposition 3.3,  $\psi(t, 0) \leq g(t)$ ,  $0 \leq t < T$ . In view of (i) and eq. (3.4), the left side of the inequality is equal to zero. This yields  $g \equiv 0$ . Now by Lemma 3.2 and (i) (see (3.4))

$$0 = \psi_I(t, 0) = \int_0^t R_0(\psi_I(s, 0)) ds = \gamma_0 t, \quad t \in [0, T),$$



which implies  $\gamma_0 = 0$  and hence (ii).

(ii)  $\Rightarrow$  (i): By Lemma 3.2,  $g := \psi_I(\cdot, 0)$  is a solution of (3.3) with  $g(0) = 0$  and values in  $\mathbb{R}_+^m$ . Assumption (ii) implies  $\psi_I(\cdot, 0) \equiv 0$ . Now  $\gamma_0 = R_0(0) = 0$  as well as  $\psi_0(t_0, 0) = 0$  and (3.1) yield  $\psi_0(\cdot, 0) \equiv 0$ . Hence (3.4) holds and (i) follows.

Finally, we show that either (i) or (ii) implies  $(\gamma_1, \dots, \gamma_m) = 0$ . Note that by Definition 2.1 we have  $\gamma_{m+1} = \dots = \gamma_d = 0$ . From (3.2) for  $u_I = 0$  and from (3.4) it follows that  $0 = R_j(0) \cdot t$  and hence  $R_j(0) = \gamma_j = 0$  for all  $1 \leq j \leq m$ .  $\square$

**Remark 3.5.** (i) By Definition 2.1,  $R_0(0) = 0$ ,  $R(0) = 0$  is equivalent to  $\gamma = 0$ . This means that the infinitesimal generator of the associated Markovian semi-group has zero potential, see [8, Equation (2.12)]. If an affine process with  $\gamma = 0$  fails to be conservative, then it must have state-dependent jumps.

- (ii) The comparison results established in Appendix A are the major tool for proving Proposition 3.3. They are quite general and therefore allow for a similar characterization of conservativeness of affine processes on geometrically more involved state-spaces (as long as they are proper closed convex cones). In particular, such a characterization can be derived for affine processes on the cone of symmetric positive semidefinite matrices of arbitrary dimension, see also [5, Remark 2.5].
- (iii) Conservativeness of  $(X, \mathbb{P}_x)_{x \in D}$  and uniqueness for solutions of the ODE (3.3) can be ensured by requiring

$$\int_{D \setminus \{0\}} (|\xi_k| \wedge |\xi_k|^2) \kappa_j(d\xi) < \infty, \quad 1 \leq k, j \leq m, \quad (3.5)$$

as in [8, Lemma 9.2], which implies that  $R_I(\cdot, 0)$  is locally Lipschitz continuous on  $\mathbb{R}_+^m$ .

- (iv) If  $m = 1$ , conservativeness corresponds to uniqueness of a one dimensional ODE and can be characterized more explicitly: [8, Corollary 2.9], [10, Theorem 4.11] and Theorem 3.4 yield that  $(X, \mathbb{P}_x)_{x \in D}$  is conservative if and only if either (3.5) holds or

$$\int_{0-} \frac{1}{R_1(u_1, 0)} du_1 = -\infty, \quad (3.6)$$

where  $\int_{0-}$  denotes an integral over an arbitrarily small left neighborhood of 0.

The sufficient condition (3.5) from [8, Lemma 9.2] is easy to check in applications, since it can be read off directly from the parameters of  $X$ . However, the following example shows that it is not necessarily satisfied for conservative affine processes. This example is somewhat artificial and constructed so that the moment condition (3.5) fails but the well-known Osgood condition (3.6) does not. While it is possible to extend the example in several directions (infinite activity, stable-like tails instead of discrete support, multivariate processes, etc.), we chose to present the simplest version in order to highlight the idea.

**Example 3.6.** Define the measure

$$\mu := \sum_{n=1}^{\infty} \frac{\delta_n}{n^2},$$

where  $\delta_n$  is the Dirac measure supported by the one-point set  $\{n\}$ . Then we have

$$\beta_1 := \int_0^{\infty} h(\xi) d\mu(\xi) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore the parameters  $(\alpha, \beta, \gamma, \kappa)$  defined by

$$\alpha = (0, 0), \quad \beta = (0, \beta_1), \quad \gamma = (0, 0), \quad \kappa = (0, \mu)$$

are admissible in the sense of Definition 2.1. Denote by  $(X, \mathbb{P}_X)_{X \in \mathbb{R}_+}$  the corresponding affine process provided by Theorem 2.2. Then

$$\int_0^{\infty} (|\xi| \wedge |\xi|^2) d\mu(\xi) = \int_1^{\infty} \xi d\mu(\xi) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which violates the sufficient condition (3.5) for conservativeness. However, we now show that the necessary and sufficient condition (ii) of Theorem 3.4 is fulfilled, which in turn ensures the conservativeness of  $(X, \mathbb{P}_X)_{X \in \mathbb{R}_+}$ . By construction,  $R_0(u) = 0$  and

$$R(u) = R_1(u) = \int_1^{\infty} (e^{u\xi} - 1) d\mu(\xi) = \sum_{n=1}^{\infty} \frac{e^{un} - 1}{n^2}. \quad (3.7)$$

Clearly,  $R(u)$  is smooth on  $(-\infty, 0)$ , and differentiation of the series on the right-hand side of (3.7) yields

$$R'(u) = \sum_{n=1}^{\infty} \frac{e^{un}}{n} \quad (3.8)$$

$$R''(u) = \sum_{n=1}^{\infty} e^{un} = \frac{e^u}{1 - e^u}. \quad (3.9)$$

By (3.9), we have  $R'(u) = -\ln(1 - e^u) + C$  and further by (3.8),  $R'(u)$  tends to zero as  $u \rightarrow -\infty$  and, therefore,  $C = 0$ . We thus obtain

$$R'(u) = -\ln(1 - e^u). \quad (3.10)$$

Since  $1 - e^u = -u + O(u^2)$ , we have  $1 - e^u \geq -u/2$  for  $u \leq 0$  small enough. Hence,

$$0 \leq R'(u) \leq -\ln\left(-\frac{u}{2}\right)$$

for  $u \leq 0$  small enough. As  $R(0) = 0$  by (3.7), it follows that

$$0 \geq R(u) = - \int_u^0 R'(u') du' \geq \int_u^0 \ln\left(\frac{-u'}{2}\right) du' = -u \ln\left(\frac{-u}{2}\right) + u \geq -2u \ln\left(\frac{-u}{2}\right) \quad (3.11)$$

for  $u \leq 0$  small enough. This implies

$$\int_{-1}^{0-} \frac{du}{R(u)} = -\infty;$$



hence  $(X, \mathbb{P}_x)_{x \in \mathbb{R}_+}$  is conservative by Remark 3.5 (iv).

#### 4. EXPONENTIALLY AFFINE MARTINGALES

We now turn to the characterization of exponentially affine martingales. Henceforth, let  $(X, \mathbb{P}_x)_{x \in D}$  be the canonical realization on  $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}_t^d)_{t \in \mathbb{R}_+})$  of a conservative, stochastically continuous affine process with corresponding admissible parameters  $(\alpha, \beta, 0, \kappa)$ .

We proceed as follows. First, we characterize the *local* martingale property and the positivity of stochastic exponentials. Since these are “local” properties, they can be read directly from the parameters of the process. Afterwards, we consider the *true* martingale property of  $\mathcal{E}(X^i)$ . Using Girsanov’s theorem, we first establish that it is *necessary* that a related affine process is conservative. Afterwards, we adapt the arguments from [15] to show that this is also a *sufficient* condition. Combined with the results of Section 3, this then characterizes the true martingale property of  $\mathcal{E}(X^i)$  in terms of uniqueness of the solution of a system of generalized Riccati equations. Finally, we adapt our Example 3.6 to construct an exponentially affine local martingale  $\mathcal{E}(X^i)$  for which the sufficient condition of [15] fails, but uniqueness of the Riccati equations and hence the true martingale property of  $\mathcal{E}(X^i)$  is assured by the Osgood condition (3.6).

We begin with the local properties. Our first lemma shows that it can be read directly from the corresponding parameters whether  $\mathcal{E}(X^i)$  is a local martingale.

**Lemma 4.1.** *Let  $i \in \{1, \dots, d\}$ . Then  $\mathcal{E}(X^i)$  is a local  $\mathbb{P}_x$ -martingale for all  $x \in D$  if and only if*

$$\int_{\{|\xi_i| > 1\}} |\xi_i| \kappa_j(d\xi) < \infty, \quad 0 \leq j \leq d, \quad (4.1)$$

and

$$\beta_j^i + \int_{D \setminus \{0\}} (\xi_i - h_i(\xi)) \kappa_j(d\xi) = 0, \quad 0 \leq j \leq d. \quad (4.2)$$

*Proof.*  $\Leftarrow$ : On any finite interval  $[0, T]$ , the mapping  $t \mapsto X_{t-}$  is  $\mathbb{P}_x$ -a.s. bounded for all  $x \in D$ . Hence it follows from Theorem 2.3 and [14, Lemma 3.1] that  $X^i$  is a local  $\mathbb{P}_x$ -martingale. Since  $\mathcal{E}(X^i) = 1 + \mathcal{E}(X^i)_- \bullet X^i$  by definition of the stochastic exponential, the assertion now follows from [13, I.4.34], because  $\mathcal{E}(X^i)_-$  is locally bounded.

$\Rightarrow$ : As  $\kappa_j = 0$  for  $j = m+1, \dots, d$  and  $X_-^j$  is nonnegative for  $j = 1, \dots, m$ , [14, Lemma 3.1] and Theorem 2.3 yield that  $\int_{\{|\xi_i| > 1\}} |\xi_i| \kappa_0(d\xi) < \infty$  and

$$\int_{\{|\xi_i| > 1\}} |\xi_i| \kappa_j(d\xi) X_-^j < \infty, \quad 1 \leq j \leq m, \quad (4.3)$$

up to a  $d\mathbb{P}_x \otimes dt$ -null set on  $\Omega \times \mathbb{R}_+$  for any  $x \in D$ . Now observe that (4.3) remains valid if  $X_-$  is replaced by  $X$ , because  $X_- = X$  holds  $d\mathbb{P}_x \otimes dt$ -a.e., for any  $x \in D$ . Setting  $\Omega_x = \{X_0 = x\}$  for some  $x \in D$  with  $x^j > 0$ , the right-continuity of  $X$  shows that there exist  $\varepsilon > 0$  and a strictly positive random variable  $\tau$  such that  $X_t^j(\omega) \geq \varepsilon$  for all  $0 \leq t \leq \tau(\omega)$  and for all  $\omega \in \Omega_x$ . Denoting the set on which (4.3) holds by

$\widetilde{\Omega}_0$ , it follows that the set  $\widetilde{\Omega}_0 \cap \llbracket 0, \tau \rrbracket \cap \Omega_x \times \mathbb{R}_+ \subset \Omega \times \mathbb{R}_+$  has strictly positive  $d\mathbb{P}_x \otimes dt$ -measure. Therefore it contains at least one  $(\omega, t)$  for which

$$\varepsilon \int_{\{|\xi_i| > 1\}} |\xi_i| \kappa_j(d\xi) \leq \int_{\{|\xi_i| > 1\}} |\xi_i| \kappa_j(d\xi) X_t^j(\omega) < \infty.$$

Hence (4.1) holds. We now turn to (4.2), which is well-defined by (4.1). Set

$$\widetilde{\beta}_j^i := \beta_j^i + \int_{D \setminus \{0\}} (\xi_i - h_i(\xi)) \kappa_j(d\xi), \quad 0 \leq j \leq d.$$

Again by [14, Lemma 3.1] and Theorem 2.3, we have

$$\widetilde{\beta}_0^i + \sum_{j=1}^d \widetilde{\beta}_j^i X_-^j = 0, \quad (4.4)$$

up to a  $d\mathbb{P}_x \otimes dt$ -null set on  $\Omega \times \mathbb{R}_+$  for all  $x \in D$ . As above, (4.4) remains valid if  $X_-$  is replaced by  $X$ . But now, using Fubini's theorem and the right-continuity of  $X$  we find that (4.4) holds for *all*  $t \geq 0$  and for all  $\omega$  from a set  $\Omega_x$  with  $\mathbb{P}_x(\Omega_x) = 1$ . For  $x = 0$  and  $t = 0$  this yields  $\widetilde{\beta}_0^i = 0$ . Next we choose  $x = e_k$  (the  $k$ -th unit-vector of the canonical basis in  $\mathbb{R}^d$ ) and  $t = 0$ . In view of  $\widetilde{\beta}_0 = 0$ , (4.4) implies  $\widetilde{\beta}_k^i = 0$ . Hence (4.2) holds and we are done.  $\square$

The nonnegativity of  $\mathcal{E}(X^i)$  can also be characterized completely in terms of the parameters of  $X$ .

**Lemma 4.2.** *Let  $i \in \{1, \dots, d\}$ . Then  $\mathcal{E}(X^i)$  is  $\mathbb{P}_x$ -a.s. nonnegative for all  $x \in D$  if and only if*

$$\kappa_j(\{\xi \in D : \xi_i < -1\}) = 0, \quad 0 \leq j \leq m. \quad (4.5)$$

*Proof.* Fix  $x \in D$  and let  $T > 0$ . By [13, I.4.61],  $\mathcal{E}(X^i)$  is  $\mathbb{P}_x$ -a.s. nonnegative on  $[0, T]$  if and only if  $\mathbb{P}_x(\exists t \in [0, T] : \Delta X_t^i < -1) = 0$ . By [13, II.1.8] and Theorem 2.3 this in turn is equivalent to

$$\begin{aligned} 0 &= \mathbb{E}_x \left( \sum_{t \leq T} 1_{(-\infty, -1)}(\Delta X_t^i) \right) \\ &= \mathbb{E}_x \left( 1_{(-\infty, -1)}(\xi_i) * \mu_T^X \right) \\ &= \mathbb{E}_x \left( 1_{(-\infty, -1)}(\xi_i) * \nu_T \right) \\ &= T \kappa_0(\{\xi \in D : \xi_i < -1\}) + \sum_{j=1}^m \kappa_j(\{\xi \in D : \xi_i < -1\}) \int_0^T \mathbb{E}_x(X_{t-}^j) dt. \end{aligned} \quad (4.6)$$

$\Leftarrow$ : Evidently, (4.5) implies (4.6) for every  $T$ .

$\Rightarrow$ : Since  $X^j$  is nonnegative for  $j = 1, \dots, m$ , (4.6) implies that  $\kappa_0(\{\xi \in D : \xi_i < -1\}) = 0$  and  $\kappa_j(\{\xi \in D : \xi_i < -1\}) \int_0^T \mathbb{E}_x(X_{t-}^j) dt = 0$  for all  $x \in D$ . As in the proof of Lemma 4.1, it follows that  $\int_0^T \mathbb{E}_x(X_{t-}^j) dt$  is strictly positive for any  $x \in D$  with  $x^j > 0$ . Hence  $\kappa_j(\{\xi \in D : \xi_i < -1\}) = 0$ , which completes the proof.  $\square$

Every positive local martingale of the form  $M = \mathcal{E}(X^i)$  is a true martingale for processes  $X^i$  with independent increments by [15, Proposition 3.12]. In general, this does not hold true for affine processes as exemplified by [15, Example 3.11], where the following *necessary* condition is violated.

**Lemma 4.3.** *Let  $i \in \{1, \dots, d\}$  such that  $M = \mathcal{E}(X^i)$  is  $\mathbb{P}_x$ -a.s. nonnegative for all  $x \in D$ . If  $M$  is a local  $\mathbb{P}_x$ -martingale for all  $x \in D$ , the parameters  $(\alpha^\star, \beta^\star, 0, \kappa^\star)$  given by*

$$\alpha_j^\star := \alpha_j, \quad 0 \leq j \leq m, \quad (4.7)$$

$$\beta_j^\star := \beta_j + \alpha_j^i + \int_{D \setminus \{0\}} (\xi_i h(\xi)) \kappa_j(d\xi), \quad 0 \leq j \leq d, \quad (4.8)$$

$$\kappa_j^\star(d\xi) := (1 + \xi_i) \kappa_j(d\xi), \quad 0 \leq j \leq d, \quad (4.9)$$

are admissible. If  $M$  is a true  $\mathbb{P}_x$ -martingale for all  $x \in D$ , the corresponding affine process  $(X, \mathbb{P}_x^\star)_{x \in D}$  is conservative.

*Proof.* The first part of the assertion follows from Lemmas 4.1 and 4.2 as in the proof of [15, Lemma 3.5]. Let  $M$  be a true martingale for all  $x \in D$ . Then for every  $x \in D$ , e.g. [4] shows that there exists a probability measure  $\mathbb{P}_x^M \ll \mathbb{P}_x$  on  $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}_t^d))$  with density process  $M$ . Then the Girsanov-Jacod-Memmi theorem as in [14, Lemma 5.1] yields that  $X$  admits affine  $\mathbb{P}_x^M$ -characteristics as in (2.5)-(2.7) with  $(\alpha, \beta, 0, \kappa)$  replaced by  $(\alpha^\star, \beta^\star, 0, \kappa^\star)$ . Since  $\mathbb{P}_x^M|_{\mathcal{D}_0} = \mathbb{P}_x|_{\mathcal{D}_0}$  implies  $\mathbb{P}_x^M(X_0 = x) = 1$ , we have  $\mathbb{P}_x^M = \mathbb{P}_x^\star$  by Theorem 2.3. In particular, the transition function  $p_t^\star(x, d\xi)$  of  $(X, \mathbb{P}_x^\star)_{x \in D}$  satisfies  $1 = \mathbb{P}_x^M(X_t \in D) = \mathbb{P}_x^\star(X_t \in D) = p_t^\star(x, D)$ , which completes the proof.  $\square$

If  $M = \mathcal{E}(X^i)$  is only a local martingale, the affine process  $(X, \mathbb{P}_x^\star)_{x \in D}$  does not necessarily have to be conservative (see [15, Example 3.11]). A careful inspection of the proof of [15, Theorem 3.1] reveals that conservativeness of  $(X, \mathbb{P}_x^\star)_{x \in D}$  is also a *sufficient* condition for  $M$  to be a martingale. Combined with Lemma 4.1 and Theorem 3.4 this in turn allows us to provide the following deterministic necessary and sufficient conditions for the martingale property of  $M$  in terms of the parameters of  $X$ .

**Theorem 4.4.** *Let  $i \in \{1, \dots, d\}$  such that  $\mathcal{E}(X^i)$  is  $\mathbb{P}_x$ -a.s. nonnegative for all  $x \in D$ . Then we have equivalence between:*

- (i)  $\mathcal{E}(X^i)$  is a true  $\mathbb{P}_x$ -martingale for all  $x \in D$ .
- (ii)  $\mathcal{E}(X^i)$  is a local  $\mathbb{P}_x$ -martingale for all  $x \in D$  and the affine process corresponding to the admissible parameters  $(\alpha^\star, \beta^\star, 0, \kappa^\star)$  given by (4.7)-(4.9) is conservative.
- (iii) (4.1) and (4.2) hold and  $g = 0$  is the only  $\mathbb{R}_+^m$ -valued local solution of

$$\partial_t g(t) = R_I^\star(g(t), 0), \quad g(0) = 0, \quad (4.10)$$

where  $R^\star$  is given by (2.4) with  $(\alpha^\star, \beta^\star, 0, \kappa^\star)$  instead of  $(\alpha, \beta, \gamma, \kappa)$ .

*Proof.* (i)  $\Rightarrow$  (ii): This is shown in Lemma 4.3.

(ii)  $\Rightarrow$  (iii): This follows from Lemma 4.1 and Theorem 3.4.

(iii)  $\Rightarrow$  (i): By (4.1), (4.2) and Lemma 4.2, Assumptions 1-3 of [15, Theorem 3.1] are satisfied. Since we consider time-homogeneous parameters here, Condition 4 of [15, Theorem 3.1] also follows immediately from (4.1). The final Condition 5 of [15, Theorem 3.1] is only needed in [15, Lemma 3.5] to ensure that a semimartingale with affine characteristics relative to  $(\alpha^*, \beta^*, 0, \kappa^*)$  exists. In view of the first part of Lemma 4.3, Theorem 3.4 and Theorem 2.3 it can therefore be replaced by requiring that 0 is the unique  $\mathbb{R}_+^m$ -valued solution to (4.10). The proof of [15, Theorem 3.1] can then be carried through unchanged.  $\square$

**Remark 4.5.** (i) In view of [15, Lemma 2.7],  $\tilde{M} := \exp(X^i)$  can be written as  $\tilde{M} = \exp(X_0^i) \mathcal{E}(\tilde{X}^i)$  for the  $d+1$ -th component of the  $\mathbb{R}_+^m \times \mathbb{R}^{n+1}$ -valued affine process  $(X, \tilde{X}^i)$  corresponding to the admissible parameters  $(\tilde{\alpha}, \tilde{\beta}, 0, \tilde{\kappa})$  given by  $(\tilde{\alpha}_{d+1}, \tilde{\beta}_{d+1}, \tilde{\kappa}_{d+1}) = (0, 0, 0)$  and

$$(\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\kappa}_j(G)) := \left( \begin{pmatrix} \alpha_j & \alpha_j^i \\ \alpha_j^i & \alpha_j^{ii} \end{pmatrix}, \begin{pmatrix} \beta_j \\ \tilde{\beta}_j^{d+1} \end{pmatrix}, \int_{D \setminus \{0\}} 1_G(\xi, e^{\xi_i} - 1) \kappa_j(d\xi) \right)$$

for  $G \in \mathcal{B}^{d+1}$ ,  $j = 0, \dots, d$ , and

$$\tilde{\beta}_j^{d+1} = \beta_j^i + \frac{1}{2} \alpha_j^{ii} + \int_{D \setminus \{0\}} (h_i(e^{\xi_i} - 1) - h_i(\xi)) \kappa_j(d\xi).$$

This allows to apply Theorem 4.4 in this situation as well.

- (ii) Theorem 4.4 is stated for the stochastic exponential  $\mathcal{E}(X^i)$  of  $X^i$ , that is, the projection of  $X$  to the  $i$ -th component. It can, however, also be applied to the stochastic exponential  $\mathcal{E}(A(X))$  of a general affine functional  $A : D \rightarrow \mathbb{R} : x \mapsto p + Px$ , where  $p \in \mathbb{R}$  and  $P \in \mathbb{R}^d$ . To see this, note that it follows from Itô's formula and Theorem 2.3 that the  $\mathbb{R}_+^m \times \mathbb{R}^{n+1}$ -valued process  $Y = (X, A(X))$  is affine with admissible parameters  $(\tilde{\alpha}, \tilde{\beta}, 0, \tilde{\kappa})$  given by  $(\tilde{\alpha}_{d+1}, \tilde{\beta}_{d+1}, \tilde{\kappa}_{d+1}) = (0, 0, 0)$  and

$$\tilde{\alpha}_j = \begin{pmatrix} \alpha_j & \alpha_j P \\ P^\top \alpha_j & P^\top \alpha_j P \end{pmatrix}, \quad \tilde{\beta}_j = \begin{pmatrix} \beta_j \\ P^\top \beta_j + \int (h(P^\top x) - P^\top h(x)) \kappa_j(dx) \end{pmatrix},$$

as well as

$$\tilde{\kappa}_j(G) = \int_{D \setminus \{0\}} 1_G(x, P^\top x) \kappa_j(dx) \quad \forall G \in \mathcal{B}^{d+1},$$

for  $j = 0, \dots, d$ . Therefore one can simply apply Theorem 4.4 to  $\mathcal{E}(Y^{d+1})$ .

- (iii) Conservativeness of  $(X, \mathbb{P}_x^\star)_{x \in D}$  and uniqueness for solutions of ODE (3.3) can be ensured by requiring the moment condition (3.5) for  $\kappa_j^\star$ . The implication (iii)  $\Rightarrow$  (i) in Theorem 4.4 therefore leads to the easy-to-check sufficient criterion [15, Corollary 3.9] for the martingale property of  $M$ .
- (iv) By Remark 3.5 (iv) we know that in the case  $m = 1$ ,  $(X, \mathbb{P}_x^\star)_{x \in D}$  is conservative if and only if either (3.5) holds for  $\kappa_j^\star$  or equation (3.6) holds for  $R_1^\star$ . Together with Remark (i), this leads to the necessary and sufficient condition for the martingale property of ordinary exponentials  $\exp(X^i)$  obtained in [18, Theorem 2.5].

We conclude by providing an example of an exponentially affine local martingale for which the sufficient conditions from [15] cannot be applied. Our main Theorem 4.4, however, shows that is indeed a true martingale. This process is based on the one in Example 3.6 and therefore again somewhat artificial. Various extensions are possible, but we again restrict ourselves to the simplest possible specification here.

**Example 4.6.** Consider the  $\mathbb{R}_+ \times \mathbb{R}$ -valued affine process  $(X^1, X^2)$  corresponding to the admissible parameters

$$\alpha = (0, 0, 0), \quad \beta = (0, \beta_1, 0), \quad \gamma = (0, 0, 0), \quad \kappa = (0, \kappa_1, 0),$$

where

$$\begin{pmatrix} \beta_1^1 \\ \beta_1^2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{1}{(1+n)n^2} \\ \sum_{n=1}^{\infty} \frac{1-n}{(1+n)n^2} \end{pmatrix} \quad \text{and} \quad \kappa_1 = \sum_{n=1}^{\infty} \frac{\delta_{(n,n)}}{(1+n)n^2},$$

for the Dirac measures  $\delta_{(n,n)}$  supported by  $\{(n, n)\}, n \in \mathbb{N}$ . Since  $X^2$  has only positive jumps,  $\mathcal{E}(X^2)$  is positive. Moreover, it is a local martingale by Lemma 4.1, because

$$\int_{\{|\xi_2| > 1\}} |\xi_2| \kappa_1(d\xi) = \sum_{n=1}^{\infty} \frac{1}{(1+n)n} < \infty$$

and  $\beta_1^2 + \int_0^{\infty} (\xi_2 - h_2(\xi_2)) \kappa_1(d\xi) = 0$ . Note that [15, Corollary 3.9] is not applicable, because

$$\int_{\{|\xi_2| > 1\}} \xi_1 (1 + \xi_2) \kappa_1(d\xi) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

However, by Theorem 4.4 and Remark 4.5(iii),  $\mathcal{E}(X^2)$  is a true martingale, since we have shown in Example 3.6 that (3.6) is satisfied for

$$R_1^*(u_1, 0) = \sum_{n=1}^{\infty} \frac{e^{u_1 n} - 1}{n^2}.$$

#### APPENDIX A. ODE COMPARISON RESULTS IN NON-LIPSCHITZ SETTING

Let  $C$  be a closed convex proper cone with nonempty interior  $C^\circ$  in a normed vector space  $(E, \|\cdot\|)$ . The partial order induced by  $C$  is denoted by  $\leq$ . For  $x, y \in E$ , we write  $x \ll y$  if  $y - x \in C^\circ$ . We denote by  $C^*$  the dual cone of  $C$ . Let  $D_g$  be a set in  $E$ . A function  $g: D_g \rightarrow E$  is called *quasimonotone increasing*, in short *qmi*, if for all  $l \in C^*$ , and  $x, y \in D_g$

$$(x \leq y, l(x) = l(y)) \Rightarrow (l(g(x)) \leq l(g(y))).$$

The next lemma is a special case of Volkmann's result [24, Satz 1].

**Lemma A.1.** *Let  $0 < T \leq \infty$ ,  $D_f \subset E$ , and  $f: [0, T) \times D_f \rightarrow E$  be such that  $f(t, \cdot)$  is qmi on  $D_f$  for all  $t \in [0, T)$ . Let  $\zeta, \eta: [0, T) \rightarrow D_f$  be curves that are continuous on  $[0, T)$  and differentiable on  $(0, T)$ . Suppose  $\zeta(0) \gg \eta(0)$  and  $\dot{\zeta}(t) - f(t, \zeta(t)) \gg \dot{\eta}(t) - f(t, \eta(t))$  for all  $t \in (0, T)$ . Then  $\zeta(t) \gg \eta(t)$  for all  $t \in [0, T)$ .*

A function  $g : [0, T) \times D_g \rightarrow E$  is called *locally Lipschitz*, if for all  $0 < t < T$  and for all compact sets  $K \subset D_g$  we have

$$L_{t,K}(g) := \sup_{0 < \tau < t, x, y \in K: x \neq y} \frac{\|g(\tau, x) - g(\tau, y)\|}{\|x - y\|} < \infty$$

where  $L_{t,K}(g)$  is usually called the Lipschitz constant.

We now use Lemma A.1 to prove the following general comparison result.

**Proposition A.2.** *Let  $T$ ,  $D_f$ , and  $f$  be as in Lemma A.1. Suppose, moreover, that  $D_f$  has a nonempty interior and  $f$  is locally Lipschitz on  $[0, T) \times D_f^\circ$ . Let  $\zeta, \eta : [0, T) \rightarrow D_f$  be curves that are continuous on  $[0, T)$ , differentiable on  $(0, T)$ , and satisfy the conditions*

- (i)  $\eta(t) \in D_f^\circ$
- (ii)  $\dot{\zeta}(t) - f(t, \zeta(t)) \geq \dot{\eta}(t) - f(t, \eta(t))$
- (iii)  $\zeta(0) \geq \eta(0)$

for all  $t \in [0, T)$ . Then  $\zeta(t) \geq \eta(t)$  for all  $t \in [0, T)$ .

*Proof.* Fix  $t_0 \in [0, T)$ . Since  $\eta$  is continuous, the image  $S$  of the segment  $[0, t_0]$  under the map  $\eta$  is a compact subset of  $D_f^\circ$ . Let  $\delta > 0$  be such that the closed  $\delta$ -neighborhood  $S_\delta$  of  $S$  is contained in  $D_f^\circ$ . By the local Lipschitz continuity of  $f$  on  $D_f^\circ$ , there exists a constant  $L > 0$  such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad (\text{A.1})$$

for any  $t \in [0, t_0]$  and  $x, y \in S_\delta$ . Let  $c \in C^\circ$  be such that  $\|c\| = 1$  and let  $d_c$  denote the distance from  $c$  to the boundary  $\partial C$  of  $C$ . For  $\varepsilon > 0$ , we set  $h_\varepsilon(t) := \varepsilon e^{2Lt/d_c} c$ . If  $\varepsilon \leq e^{-2Lt_0/d_c} \delta$ , then  $\eta(t) - h_\varepsilon(t) \in S_\delta$  for any  $t \in [0, t_0]$ , and (A.1) gives

$$\|f(t, \eta(t) - h_\varepsilon(t)) - f(t, \eta(t))\| \leq L\|h_\varepsilon(t)\|, \quad t \in [0, t_0]. \quad (\text{A.2})$$

Since  $C$  is a cone, the distance from  $Lh_\varepsilon(t)/d_c$  to  $\partial C$  is equal to  $L\varepsilon e^{2Lt/d_c} = L\|h_\varepsilon(t)\|$ . In view of (A.2), it follows that

$$Lh_\varepsilon(t)/d_c \geq f(t, \eta(t) - h_\varepsilon(t)) - f(t, \eta(t))$$

and hence

$$-\dot{h}_\varepsilon(t) = -2Lh_\varepsilon(t)/d_c \ll f(t, \eta(t) - h_\varepsilon(t)) - f(t, \eta(t)), \quad t \in [0, t_0], \quad (\text{A.3})$$

for  $\varepsilon$  small enough. This implies that

$$\dot{\zeta}(t) - f(t, \zeta(t)) \geq \dot{\eta}(t) - f(t, \eta(t)) \gg \dot{\eta}(t) - \dot{h}_\varepsilon(t) - f(t, \eta(t) + h_\varepsilon(t)).$$

Applying Lemma A.1 to the functions  $\zeta(t)$  and  $\eta(t) + h_\varepsilon(t)$  yields  $\zeta(t) \gg \eta(t) + h_\varepsilon(t)$ , for all  $t \in [0, t_0]$ . Now letting  $\varepsilon \rightarrow 0$  yields the required inequality for all  $t \in [0, t_0]$ . This proves the assertion, because  $t_0 < T$  can be chosen arbitrarily.  $\square$

If we consider the differential equation

$$\dot{\xi} = f(t, \xi(t)), \quad \xi(0) = u \in D_f, \quad (\text{A.4})$$

Proposition A.2 allows the following immediate conclusion, which is the key tool for proving Proposition 3.3 and in turn Theorem 3.4.

**Corollary A.3.** *Let  $T$ ,  $D_f$  and  $f$  be as in Lemma A.2. Suppose further that equation (A.4) gives rise to a global solution  $\psi^\circ(t, u): \mathbb{R}_+ \times D_f^\circ \rightarrow D_f^\circ$ . Let  $u_2 \in D_f^\circ$  and let  $\xi: [0, T) \rightarrow D_f$  be a solution of (A.4) such that  $\xi(0) = u_1 \geq u_2$ . Then  $\xi(t) \geq \psi^\circ(t, u_2)$ , for all  $t \in [0, T)$ .*

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